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A New View of Mathematics Will Help Mathematics Teachers

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Abstract

For many people mathematics is something like a very huge and impressive building. It has a given structure with lots of levels and rooms. For many people this structure and therefore mathematics itself is independent from society, culture and history. It exists and mathematicians try to recover (not: to construct!) new parts of it. From this point of view mathematics is often seen as a lifeless and strange thing and not as a living construct of human beings.

Many mathematics teachers argue that they can't change their way of teaching because they see mathematics from this dominant point of view and think that mathematics will not allow changes. Asking what this means they say that mathematics is something independent from them with a fixed structure. Therefore they have to teach little parts of mathematics (often concentrated on the correct use of algorithms) in a fixed sequence. Changing the sequence or leaving out a part seems to be not allowed. So they are not happy with mathematics but they see no way to change mathematics and therefore no way to change their teaching (perhaps except in some minor important methodical aspects).

I think there is a "way out" if mathematics is seen as a social construct. Is this view correct? A new look at the history of mathematics proves that the history of mathematics in the last 200 years looks like a very good example of applying a sociological theory to make a new interpretation of what happens. In more provocative words: If the sociological theory I apply to make my interpretation of the history of mathematics had existed 200 years ago one could think that the mathematicians tried to prove that the sociological theory is correct by forming the history of mathematics in the way the theory "wants". If teachers try to share this view they will be able to recognize that mathematicians decided what "mathematics" is. I hope this will motivate teachers to make more individual and pedagogical decisions on what and how they teach.

Introduction

In my opinion it is very important for our society to change the image of mathematics, since many studies show that the bad image of mathematics has not changed in past hundred years.

On the one hand many adults remember their experience of learning mathematics at school or university as a bad and stressful situation. They did not understand why they should learn to handle all these difficult algorithms. Many teachers have not answered their important "Why?" - question satisfactorily. Teachers often tell their students: "You will understand this later on, after you have learned mathematics!" Many mathematicians as well as educators and teachers of mathematics have criticised this – for example, Felix Klein about a hundred years ago. But the situation has not changed. If we read Klein today (see Klein, 1926, and his contributions to the "Meraner Reform" analysed by Inhetveen, 1976) we have the feeling that he is talking about today's mathematics lessons.

On the other hand *"too few people recognize, that the high technology so celebrated today is essentially a mathematical technology",* according to Edward E. David, former president of Exxon Research and Development (David, 1984, p. 142). Mathematics is really a powerful force enhancing technological development and changing society and the lives of everyone:

- Mathematics is the basis of the new technologies, since mathematical algorithms are included in all computer software, and computer hardware is materialised mathematical logic.
- Increasingly efficient computers with a great variety of software also play an important part in contributing to the growing influence of mathematics, not only indirectly through other sciences, but also directly through the mathematical models integrated in standard software.
- Mathematical methods and ways of thinking are being used in an ever-increasing number of scientific disciplines (Who is not using statistical methods nowadays?) and, as a result, in more and more spheres of life.
- Mathematics helps to plan the future by constructing models and simulating different development strategies in economics and politics (see Maasz & Schloeglmann, 1988).

The public opinion about mathematics and its real influence on life and society do not match up. Not understanding the sense of learning and the correct ways of applying mathematics reduces the chances of many people. Changing the teaching methods of mathematics would be very helpful in this respect, but it is not enough. Looking at mathematics from a new point of view would open new chances for adults and for the society. What I intend to do with my contribution is to open a new view on mathematics itself. This view might be helpful in changing people's attitude towards mathematics, too.

The Sociological Theory used in this Article

The starting point of my analysis of mathematics from a sociological point of view is the selection of a suitable sociological theory. I think that a version of Niklas Luhmann's theory of social systems would be very useful but this is not the right place to explain this decision by comparing his and other sociological theories. Luhmann (1970, 1984) has developed his theory of social systems over many years and has changed several aspects and definitions. This has lead to an ongoing discussion in sociology and has introduced developments in other disciplines (see, for example, http://en.wikipedia.org/wiki/systems_theory). This paper is structured by my understanding of Luhmann's theory (see Maasz, 1985).

The historical background

If we read Bourbaki we get the impression that there is one and only one mathematics and the history of mathematics is the history of important persons who discovered parts of it. If there is any social influence on mathematics in this view it is concentrated on one aspect: How rich is the society? How much research is funded? This decides how fast new mathematics is discovered. The results of research into mathematics are fixed in Plato's world of ideas before the research starts and it is independent from us. This image of mathematics is one important reason for the authority of mathematics – there is nothing to construct.

I should point out that I am not writing a new history of mathematics. I draw on several historical facts to collect the data I need to illustrate that the history of mathematics has happened as if it follows Luhmann's theory. Or in other words: Luhmann's theory is a good key to understanding what happened over the last two centuries and why. This sociological view gives us much more understanding than a philosophical approach.

In this article I focus on historical facts that happened in the centre of Europe, especially in Preussen (i.e., Prussia). This might appear Eurocentric but I think that many of the most important decisions about mathematics with ongoing consequences really happened then and there.

Links to Adults Learning Mathematics

This article is not intended as course material for a typical group of adults learning mathematics. The argumentation proposed here (for details see Maasz, 1988) is part of an academic debate about mathematics and its relation to society. This debate should be part of studying mathematics at university level. This article is written to suggest to teachers that there is a debate about mathematics and society. This might motivate them to read more about it, or at least to feel more free to change their emotional relation to mathematics, as a precondition to help to change the image of mathematics that adults learning mathematics in their courses have. I do not expect that this article answers all questions arising from reading it. But I hope that we may start to have a discussion about it!

Let us now begin with a look at the situation of mathematics in the 19th century. I want to point out that in this century mathematics was a part of natural sciences, personally and functionally closely connected to physics, astronomy, mechanics and geography. In the 19th century the basic structure of this social system was built up, but the functional differentiation did not happen. Mathematics became a relatively autonomous subsystem of the sciences with its own rules for accepted "truth" in the 20th century.

Looking at the History: The Building up of System Structures to Prepare for the Birth of the Social Subsystem "Mathematics"

In his analysis of science as a social system Luhmann points out the importance of a good solution for the problems of obtaining and processing information. These problems were solved by building up structures of communication including locations to meet informants; to find literature, to exchange information and ideas, to teach and to learn. Such locations were the reformed universities and the academies of 200 years ago.

In addition to this, other structures were useful to exchange information with colleagues in other locations. In earlier centuries scientists wrote letters and travelled to visit colleagues or went to places like monasteries, where documents were stored. It is well known how important letters and visits were for mathematicians like Bernoulli, Fermat, Leibniz, Newton, Euler, and Gauss. In the 19th century new types of communication were institutionalised: Mathematicians founded journals and national organisations like the *Deutsche Mathematiker Vereinigung* (DMV) (i.e., Society of German Mathematicians). Mathematicians started to edit mathematical journals like the *Crelles Journal*, the *Mathematische Annalen* (i.e., Mathematical Yearbook), edited by A. Clebsch and C. Neumann, the *Yearbooks on the Progress in Mathematic*s. In the beginning these journals were filled with papers from different subjects like mathematics, mechanics, physics, geography, engineering, technical sciences (related to mathematics), and others. For many years the journals were not specialized in mathematics or parts of mathematics like they are today.

This is important because a Robinson Crusoe cannot build a social system. Systems are built by a lot of social interactions that constitute a continuous social activity. Inner structures of a social system are a better way to make sure that such activities are not lost or wasted. The development and the quality of science as a social system very much depends on the quality and density of information exchange. In earlier centuries the main problem of today – which is called "information overload" caused by the electronic communication infrastructure and the use we make of it – was not known. Mathematicians had great difficulty getting information about the results of their colleagues' work from anywhere else, even from a neighbouring town. Coming together at locations, attending conferences, and reading journals were important steps towards making communication more effective.

I will now summarize some aspects of this development. The first aspect seems simple: A social system needs persons who are qualified to take part in the specific interaction. Mathematics has always needed and still needs an organised process of teaching and learning mathematics. Or, in other words, there is no social system "mathematics" without mathematicians.

About 200 years ago a typical university consisted of faculties like theology, medicine, philosophy and law. Natural sciences and, as a component, mathematics were located in the philosophy faculty. Only very few people were able to study at that time. Students needed wealthy parents or another sponsor to pay for their studies. Only very few students attended lectures about mathematics. It is difficult to find out exact numbers because the statistical data is not complete. Official statistics have been collected since 1830. Lorey studied the development of mathematics at the universities in the 19th century (see Lorey, 1913, 1916). According to him the first “studiosus mathematicus” was mentioned around 1750. This is odd, because in those days it was not possible to study mathematics. Learning mathematics, namely basic algebra and geometry, was part of the general education in teacher training at the philosophy faculty for future teachers of a “Gymnasium” (i.e., Grammar school). Lectures on higher level mathematics at universities started about 100 years later – as explained below. If mathematicians gave lectures about their own research, they talked about it in a more private atmosphere, the so-called *privatissima*.

Lorey analysed statistical data about those university students who passed the state examination in the subject Mathematics and Natural Science in Prussia in the years 1839-1913. He came to the conclusion that: “We see since 1839 and at the beginning slow and after 1870-1885 rapid increase of numbers up to 160 candidates”[1] (Lorey, 1916, p. 22). In the last years of the 19th century this number was declining to the level of about 1840 (that means roughly 25 per year) and in the first years of the 20th century the number was rising to 300 candidates per year. Lorey reports on the connection of mathematics and natural science as a subject of examinations. Since 1899 chemistry and biology were separate subjects for the state examination. Physics and mathematics were closely connected and not separated in the official statistical data.

It is also difficult to find exact data on the numbers of university teachers of mathematics or mathematicians at universities. Lorey wrote that these professors were often called professor of astronomy or physics. A so-called professor of mathematics came later in the 19th century (see Lorey, 1916, p. 17). We know this fact very well from the biographies of famous mathematicians and scientists I. Many of them made important contributions to mathematics and physics, or astronomy, mechanics, or other subjects. Some were working as engineers, some as teachers at grammar schools, some were working in administration, economics, medicine, and so forth. From earlier times we know that philosophers (like Descartes) or lawyers (like Fermat) or others with several jobs (like Leibniz) worked as mathematicians, too. Hilbert gave lectures in physics and mechanics. The pure mathematician or the applied mathematician who specialized in a small part of mathematics as a full time job came later.

Looking back at the 19th century two tendencies can be recognized: On the one hand we have more and more – but all together very few - mathematicians (i.e., researchers at universities and outside universities) and, on the other hand, we have an increasing number of persons who were concentrating on the part of science we call mathematics today. An important number of research results or published papers on mathematics were produced by mathematicians outside universities. Weierstraß, for example, was a teacher at a gymnasium (grammar school) for many years and Kronecker did not work as a teacher at a gymnasium or university at all (see Ferber, 1956).

Another important group and resource for the growing mathematical knowledge were assistants and post-graduate students. After long disputes with other faculties, the departments of natural science including mathematics gained the right to promote students. Dissertations (doctoral theses) and later habilitations (postdoctoral theses) became a source of new knowledge and successful research. Some mathematical dissertations and habilitations became milestones for mathematics and for the career of the authors as famous mathematicians.

From today’s point of view this structure seems to be very clear. But 200 years ago it was new. The new type of university offered the chance to build groups of mathematicians: a full professor and his assistant professors and a group of students of mathematics working together. In the 19th century we see a development from a single professor and his group to a department of mathematics with several specialized professors and other researchers.

The Development from Single Lectures to a Regular Course of Mathematics – Another Precondition for Mathematics to Become a Social System

In the 18th century lectures on higher mathematics were very rare. Some professors liked to give such lectures about their research and some did not. The research mathematicians were autodidacts or had a mathematician as a private teacher. Today the situation is completely different. Perhaps there are some mathematicians who are autodidacts but the majority of the research mathematicians have studied mathematics at university. They have had to study all the courses prescribed in the curriculum. Graduation, dissertation thesis, postdoctoral activities, habilitation, and an appointment to become professor, are typical steps of a career as a research fellow or teaching mathematician. Some of the steps from the situation in the past to the present situation should be mentioned because they have contributed to mathematics becoming a relatively autonomous social system.

After 1830 a very important development was a new type of university structure called "mathematical seminar". The first seminars were founded in Königsberg (by C.G.J. Jacobi and F. Neumann in 1834; see Koch, 1839, 1840) and later in Berlin by K.H. Schellbach in 1861. What type of structure was a "seminar"? It was a public institution connected to the university. It should give the students of mathematics an orientation for their studies. A seminar's principal purpose was that students could learn enough mathematics during the first few semesters. As members of the seminar the students were guided to become well trained mathematicians and to become able to do their own research. Schubring (1983, p. 23) explains the importance of these seminars for the development of mathematics in three aspects:

- At the beginning of the 19th century students were not used to studying an academic subject like today. Studying rather meant learning to ride, learning to fence and learning upper class behaviour rather than learning mathematics. The seminar was a way to concentrate the activities of students on studying mathematical topics.
- The seminar had a budget to buy books and to offer a place where students had access to mathematical literature. This was much better than the usual equipment of university libraries.
- A seminar was connected with an obligation for professors: They had to give regular lectures for students, not only well-paid "privatissima".

The next development was the transformation of seminars into departments. A department is a part of the university with a budget for positions such as professors or other staff for teaching and research. Whereas a seminar had been an exclusive activity for a few students only, a department is responsible for the teaching of all students of mathematics. This new and greater task created more teaching jobs for mathematicians. Becoming an assistant professor is a good starting point in an academic career. We see a parallel development in the increase of the quality of teacher education and the institutionalisation of careers for mathematicians at the university. People could start to plan a career although such a plan included a lot of risks – as it does today.

The second half of the 19th century was a very productive time for pure mathematics. The training of students of mathematics was limited to becoming teachers at a gymnasium (grammar school); they could not become engineers then. A diploma in mathematics was only established in Germany as late as, 1942; a diploma in technical mathematics about 20 years earlier. This had a striking consequence for the type of mathematics which was taught at university. Only since then has applied mathematics found its way into the curricula beside pure mathematics.

As will be shown later, this led to conflicts with natural scientists. The important social aspect is that before the introduction of applied mathematics into the departments of mathematics the students had been trained in pure mathematics. These mathematicians had fewer problems than applied mathematicians in saying "I'm a mathematician," and not "I'm a natural scientist with major activities in mathematics". This is one root of the separation of mathematics from the natural sciences.

The Development of an (Inter-)National Mathematical Community

As mentioned before, the most important part of a social system "science" is communication. Its existence depends on an optimal infrastructure for communication. It was necessary to build up structures where results of research could be exchanged. In the 19th century an interesting development took place in this area. At the beginning of the 19th century mathematicians and other natural scientists started to found associations; at the end of the century national and international associations of mathematicians were founded separately from natural sciences.

The first organisations were local groups. Lorey reports that the Mathematical Society in Hamburg was founded in 1690 (Lorey, 1916, p. 221). Many other similar societies were founded in different university cities. Lorey describes the path from these local organisations to the foundation of the Societies like the DMV (founded in 1890) as a long process including writing letters, travelling, visiting colleagues, invitations to lectures at other universities or local organisations, and meetings of small groups of mathematicians. One of these meetings was the starting point of the journal *Mathematische Annalen* (Mathematical Annals).

Another point on the way was the separation from other organizations. In 1822 the *Gesellschaft Deutscher Naturforscher und Ärzte* (Society of German Natural Researchers and Medical Doctors) was founded. In the year 1843 a section *Mathematics and Astronomy* was started. During the annual meeting of the *Gesellschaft Deutscher Naturforscher und Ärzte* in Heidelberg the *Deutsche Mathematiker – Vereinigung* (DMV) was founded in 1890 (see Lorey, 1916, p. 213; for more information about the importance and history of the *Gesellschaft Deutscher Naturforscher und Ärzte* see Pfetsch, 1974, & Inhetveen, 1976, pp. 101–110).

Lorey noted some drawbacks on the way to a national organisation. Sometimes it was difficult to convince the mathematicians, some of whom were a little narrow-minded and kept their distance from such national questions. A meeting in a south German university town could not take place because the mathematicians living there had decided to leave the city during the time when the other mathematicians were to come.

Two persons are identified as being responsible for the success of the foundation process: Georg Cantor and Felix Klein. Lorey thinks that Cantor undertook the main efforts. He cites a *Laudatio* held by A. Gutzmer for the celebration of Cantor's 70th birthday (see Lorey, 1916, p. 215). Tobies collected documents that prove that Klein played a main role in the background to the process of the foundation of the *Deutsche Mathematiker-Vereinigung* (see Tobies, 1985, 1989).

Summary of 19th Century Developments

I have summarized some aspects of the development of mathematics in the 19th century to point out how the number of mathematicians was growing and how an effective communication structure was built up. At the end of the 19th century there were several universities with mathematics departments, a growing number of mathematicians working inside and outside of universities, an organisation of mathematicians, and many mathematical journals – some of which were very important. Altogether mathematics took a lot of important steps towards becoming a social subsystem, but it was still an integral part of the social system "natural sciences including mathematics" because these disciplines were not clearly separated.

Now I want to have a look at the discussion on the truth and relevance of mathematical theory. These debates were starting points to a separation, but the separation itself took place later on. Why? Luhmann (1970) says that the main reason for the differentiation of social systems is a different function. This means for scientific systems that there must be a difference in the way the truth of a new theory is proved and accepted. The function of scientific social systems is to separate true and not true theories (whatever "true" means).

Starting Points of the Functional Differentiation

A long time ago the development of mathematics was hampered by discussions on the truth and the meaning of the results. Non-Euclidian geometry is a well known example of this. Euclid from Alexandria (365–300B.C.) wrote the *Elements*, a famous collection of mathematical theorems. He defined what we call *Euclidian Geometry*. The fifth postulate said that in a plane there was one and only one straight line parallel to a given straight line through each point that was not an element of the straight line. Over more than 2000 years many mathematicians have tried to prove that this postulate can be deduced from the others, but they have not been successful. They only discovered other equivalent postulates.

Kant was convinced that the Euclidian geometry was the only possible geometry (see Kant, 1781). His philosophy influenced the scientists greatly; nobody published results on other types of geometry.

Gauss founded a new approach to geometry. He asked formally: What would happen, if there were more than one parallel straight line? Would it be possible to define a postulate system with this postulate, but without contradictions? About 1790 he found the non-Euclidian hyperbolic geometry but did not publish his results. He wrote letters to other scientists like Schumacher or Bessel (see Wußing, 1983) to inform them about his new ideas on geometry and his fear of publishing something contradictory to Kant's philosophy. Two mathematicians published papers on non-Euclidian geometry (Bolyai in 1832 and Lobatschewski in 1826), but both had problems in gaining acceptance. The latter wrote that the new geometry did not exist in nature but in the human mind only. Maybe it was not useful for measurement in nature, but it would open a wide area of application in both geometry and analysis (see Lobatschewski, 1899, p. 83). From today's point of view this typical formalistic argument is a common reason to publish results, but 200 years ago this was not accepted. Mathematics was located in philosophy faculties and could not act autonomously in decisions about the truth of mathematical theories. Some months before Gauss died (June 10th 1854), Riemann presented the results of his research on geometry – the theory of the *Riemann'schen Mannigfaltigkeiten* – to become a teacher at the university level. He presented many types of non-Euclidian (elliptic) geometry and an explicit rebuttal of Kant's theory on geometry.

In the 19th century a lot of work was done to find out more about analysis, especially about proofs and the foundations of analysis. Looking at the history of analysis we find some significant works written by Cauchy (1821), and about 50 years later by Weierstraß, Dedekind, and others. One basic idea was a foundation of analysis that had a quality similar to other parts of mathematics, especially arithmetic (see Cauchy, 1821; Klein, 1926; Spalt, 1981; Struik, 1976). An unexpected result of the use of arithmetic was a formalized approach, the possibility of looking at analysis from a formal point of view. Du Bois-Reymond published an example for a steady function that cannot be differentiated. He called the function *disconcerted* (Du Bois-Reymond, 1875, p. 21). Some of his colleagues called such functions *pathological*, but this expression was not accepted by many mathematicians. From my point of view this shows quite well that a formal approach often leads to unexpected results. Although the existence could be formally proved, nobody could imagine such functions or see them in reality. Looking for examples in reality is a typical method of natural scientists; today formal working mathematicians do not use this method. Their argument is that the future may show if the new theory is useful in reality. If it is not, this does not matter for acceptability today or in the future.

Cantor's results were not only "disconcerted" or "pathological". He asked for the potency of the set of natural and real numbers and showed by using formal methods that there were at least two types of infinity. Firstly, the *Kardinalzahl* (i.e.,cardinal number) of □ and *R*. Following his definition, the cardinal number of *N* is aleph 0 and the cardinal number of *R* is aleph 1 (see Cantor, 1874, republished in 1932 – his paper was published in Crelle's Journal). How are they connected? *R* is equivalent to P(N), the power set of *N*. Cantor's next question was simple: What is P(R) ? It is a set with the cardinal number aleph 2. And many questions later: Which set has the cardinal number aleph Omega that is bigger than aleph n for all n (n element of *N*)?

This was too much for his colleagues – they did not accept it. But a generation later Hilbert (1923) celebrated Cantor as the creator of a paradise for mathematicians. What had caused this change? Wang sees “The main reason, why Cantor has been so much more influential is probably his ability to renounce applications and develop the theory of sets more and more for its own sake. By generalizing and following up logical conclusions, Cantor became the founder of set theory” (Wang, 1954, p. 244).

Let me add a little *pedagogical remark* to the historical debates about geometry and analysis in the 19th century and to similar debates on research results by Graßmann (see: http://www.maths.utas.edu.au/People/dfs/Papers/GrassmannLinAlgpaper/GrassmannLinAlgpaper.html) and Hamilton (1853) – two prominent examples for the change of view in history: I think it would be a very good idea to read the original papers to understand these debates. It would open a new and deeper view inside mathematics. Most mathematicians today do not know that there was any debate in history on the acceptance of well-proven research results. Mathematicians and especially mathematics teachers should know more about mathematics!

The Crisis of Foundation of Mathematics at the Beginning of the 20th Century

Cantor and the definition of a set is a key to the next step, the historical crisis of mathematics about a hundred years ago. I will show that the most important result of this crisis was the birth of mathematics as a separate social system with a separate function: Mathematicians defined their own way of deciding the truth of mathematical research results.

Today we have a very useful synthesis of works about sets and formal logic. Papers written in English use some hundred different words and the well-known symbols of sets and logic. Mathematicians all over the world can read these papers if the words and symbols are used correctly. Thus mathematics has developed a unique international communication code. If we compare this communication code with other scientific disciplines we see enormous differences. For example books about philosophy use several thousands of non-standardised words and even some philosophers point out that it is difficult to understand what Hegel exactly meant when he used his special terms.

This useful international communication code has but one disadvantage: it is an abstract code. Applying it means filling it with sense for the area where mathematics should be used. This is not simple.

The development of this communication code was not simple, either. About 130 years ago Cantor wrote the basic ideas of the set theory. 30 years later Russell published his book *Principles of Mathematics* (Russell, 1903) including the proof for the inconsistency of the basic definition of a set. He defined the set *M* of all sets that do not include themselves. This definition leads to a contradiction: *M* cannot be an element and not an element of *M*, but *M* is a set according to Cantor’s definition of a set. This antinomy had been found several years before by Burali-Forti (1897), but Russell’s book had more readers.

At the beginning of the 20th century many mathematicians used the symbols of set theory as a useful language to write down their results. Therefore many of them wanted to know: Is this problem important for my results? Are they correct? If we are using a wrong precondition, they may be incorrect, because from false proposals everything can be deduced.

In this situation it was very helpful that two technical solutions were found and offered by Zermelo (1908) and Fraenkel (1927), and by Bernays (1930 – 1931), Gödel (1931), and von Neumann (1931). Both solutions avoid the antinomy but are formal. The authors define sets in an abstract way without any relation to reality or to philosophical reason.

From different philosophical positions several attempts were made to solve the crisis of the foundation: The existence of the mathematical objects and the proof of the truth of mathematical theories were the main questions in the philosophical debate. I can’t report this debate here (see Maasz, 1988), but I want to outline some important positions: the logicism, formalism and intuitionism.

The Logical Proposal for a Solution and its Problems

The first attempt to solve philosophical problems was made by mathematicians who were known as experts in mathematical logic like Frege, Russell, and Whitehead. Frege showed that the problems of basic definitions of a set by Cantor are language problems. Words of different “types” were mixed. (For details see the explanation of types by Frege in his theory of types – Frege, 1884, 1893, 1903).

For a long time logic was a part of philosophy. If we look at important developments we see Aristotle as a starting point, the rules of argumentation in the Middle Ages (Syllogism) and a new type of thinking about logic in the 19th century. Russell (1901) wrote that pure mathematics was invented by Boole’s book on *Laws of Thought* (1854). Frege’s logical foundations of mathematics were much stricter and clearer then everything written before (see Thiel, 1972, p. 93). But when Russell found a contradiction and told Frege about it in, 1902 – a short time before the second issue of his book should be published – this became a problem for Frege. He tried to find a way out but was not successful (see Quine, 1955, for details). Fraenkel (1927) and Hilbert (1925) report that this made him resign.

The next attempt by Russell and Whitehead avoided antinomies in their *Principia Mathematica* (Whitehead & Russell, 1910, 1912, 1913). They invented a more sophisticated theory of types but they needed two basic axioms: infinity – the existence of an unlimited totality is postulated – and reducibility: “every function of one variable is with all its values equivalent to a predicative function of the same argument” (Whitehead & Russell, 1910, p. 166) These axioms postulate what should be proved, namely the existence. In the preface to the second edition Whitehead wrote: “This axiom has a purely pragmatic justification: it leads to the desired results, and to no others. But clearly it is not the sort of axiom with which we can be satisfied” (Whitehead, Russell, 1925, p. xiv). If we assume what we want to prove we are able to prove everything. So this was a very useful and honourable work (see Rusawin, 1968, p. 243) but not the philosophical foundation of mathematics the researchers were looking for.

The Intuitionistic Proposal for the Philosophical Foundation of Mathematics

Since Plato and Aristotle we have had an ongoing debate as to whether mathematical objects are created or found by mankind (see Becker, 1927, p. 572). From the Platonist point of view mathematics exists independently from us in the area of ideas. Our task is to discover it like a sailor discovers new islands or even a continent in the ocean (see Stegmüller, 1978, p. 675). Intuitionism is a part of Constructivism, as explained in books of Aristotle, Kant, and others. Aristotle, however, believed that mathematical objects are constructed by human beings through abstraction. He collected arguments against Plato’s assumption in the first, sixth and twelfth book of his “Metaphysics” (see *Organon*, 1948).

The intuitionistic world centres around the concept of “intuition”: This means an act of gaining knowledge inside one’s own mind. Thus intuition is not a result of social or empirical activity but a result of thinking – a priori intuition – (see Brouwer, 1907, p. 179, or Heyting, 1934, p. 3). This is not to be confused with the Russian understanding of intuitionism. Markov put it this way: “I can’t accept that ‘intuitive clearness’ is a criterion for truth in mathematics because this means the total triumph for subjectivism in mathematics. This is not in accordance with the opinion that scientific research is a form of social activity” (cited in Rusawin, 1968, p. 262). The debate about the social basis of scientific constructions continues until this day.

What did intuitionism want? To save Cantor’s set theory was not among its principal targets. The intuitionists like Brouwer, Heyting or Weyl wanted to find an epistemological basis for mathematics. The method proposed was to exclude the actual infinity and the method of constructing everything. The Platonist deduction from “it is formally correctly defined” to “it exists” should not be allowed.

The result of the intuitionistic work was a new type of mathematics. It was philosophically well founded but it had additional rules and was less extended. The majority of mathematicians did not want to accept this. Hilbert wrote that the intuitionists had curtailed mathematics and had restricted research in this direction. He wrote: "Mathematics is in danger of losing a big part of its most valuable treasures" (Hilbert, 1922, p. 159). Bernays argued: "It is an unreasonable demand from philosophy to mathematics to abandon a simple and more efficient method", (Bernays, 1930–1931, p. 351). This argument shows exactly what the theory of a social system would predict: The better functionality is a main reason for opposing intuitionism. As part of a bigger social system mathematicians had to make allowances for philosophy and its demands on its methods. As a separate social system mathematics would be relatively autonomous and free to define its own methods and criteria for truth.

The intuitionistic demands were rejected. Only very few mathematicians worked on this basis. In the second half of the 20th century a group of scientists located in Erlangen (Germany) renewed the intuitionist argumentation. They showed that it was possible to prove all parts of analysis that are needed for applications in the real world (see Lorenzen, 1958; Lorenzen & Schwemmer, 1972).

The Way from the Formalistic Proposal for a Philosophical Basis by Hilbert to the Formal–Axiomatic Solution

The formalistic position consists of a variety of different approaches. Most of them are based on a Platonist point of view. Hilbert was one of the prominent representatives of formalism though he was not a pure formalist. He always asked for reasons for the mathematical theories and for their connection to reality – like in physics. Bourbaki (1971) characterizes Hilbert as an atypical formalist. According to Bourbaki, formalists believe that a formalized language has the sole function of being an unambiguous vehicle for communication. It does not matter for them whether the mathematical object exists in reality or not, as long as it is possible to write down the knowledge about it in a formalized language. But Hilbert always has believed in an objective mathematical 'truth' (see Bourbaki, 1971, p. 48).

Hilbert's aim was to definitely remove all doubts as to the certainty of mathematical conclusions (see Hilbert, 1923, p. 178). In order to reach this goal, he developed an axiomatic method. His axioms are, however, different from those of Euclid. Euclid had defined axioms in a way that they are clear and evident statements about the truth. Hilbert wilfully relinquished a relation to reality, as can be read in his book on geometry (see Hilbert, 1899). In a letter to Frege dated December 29, 1899, he explained: "If I imagine any system of objects like the system of love, law or chimney sweeper and take my axioms as relations between these objects my conclusions (for example, Pythagoras) are correct for these objects." (See also Steck, 1941).

This is a good illustration of formal thinking. Hilbert proposed that mathematicians should work with formal theories. The connection of these formal theories to reality was an additional task. This happens when the formal theory is applied in a special context or when it is filled with facts (numbers, data) from reality. If a mathematical theory presupposes formal axioms, all consecutively deduced theories are formal. All of mathematics becomes a formal theory (see Hilbert, 1925, p. 177). Mathematics becomes a collection of formally proved or deduced theories. The relation to reality is not clear and open. Hilbert did not want to stop at this level. He developed a second part of mathematics called the theory of proofs. He called this part of mathematics "Metamathematics". Metamathematics works with the proof of the other part of mathematics as objects to make sure that the axioms are free of contradictions (see Hilbert, 1923, p. 180).

From a philosophical point of view it is not sufficient to prove that axioms are free of contradictions. A proof doesn't show the existence of the axioms – if you are not a convinced Platonist. However, independent from the philosophical debate, Hilbert's program did not work. Gödel showed "that Hilbert's program is essentially hopeless" (von Neumann, 1947).

From my point of view, Hilbert's program was very successful in a non-intentional aspect – the division of mathematical work into normal work (i.e., formal work) and thinking about its foundation. Most of the mathematicians did not think about philosophical problems and foundations any more. They used the formal approach to work faster and more efficiently. Only very few mathematicians decided to do research on logical, philosophical or basic problems. For mathematics as a whole this division of work was a good method to be more successful. As is well known from other areas, the division of labour enhances the chances of increasing the output.

The Last Step on the Way to "Mathematics as a Social System": The Bourbaki Group

In 1934 a group of famous mathematicians planned to give a complete description of all parts of mathematics based on a formal axiomatic method, assuming nothing but formal axioms and deducing the rest (see Bourbaki, 1934, 1971). Cartan, one of the members of this group, wrote that they were convinced that such a building up of mathematics "ex nihilo" should be possible (Cartan, 1959, pp. 8, 12).

The Bourbaki group was very successful. Some claim that the name had been taken from a general of Napoleon's army, others maintain that the letters of the name represented members of the group. Their books gave mathematics a new basis and a very useful communication code. The main reason for the success of their work is the standardized way by which they described all aspects of mathematics. It gave an new and better overview than the "Enzyklopädie der Mathematik" which had started about 50 years earlier but was never finished (see Maasz, 1987).

Bourbaki's books opened up a wide area of new questions for mathematicians because this new view on all aspects of mathematics implied many hidden relations of different parts of mathematics. A formal view motivates one to ask more formally : "What happens if I take more dimensions or if I apply a result from one part of mathematics to another part of mathematics?". This research was not slowed down by any questions about meaning or about the connection to reality or by the need to prove the possibility of application. This formal–axiomatic method was exactly the type of method which Luhmann (1984) has described as the ideal communication code.

The Bourbaki group has clearly pointed out, that they have not been interested in philosophical questions. I do not want to argue with them about this point! (See Bourbaki, 1971, p. 39). From a philosophical point of view the philosophical problems have not been solved but suppressed (see Thiel, 1972, p. 127). I would like to add that the statement as to not being interested in philosophy is not equivalent to the absence of a philosophical position. Bourbaki is based on Plato, and Platonists believed that exists what can be defined.

Conclusion

The agreement of the majority of mathematicians to use the formal–axiomatic method proposed and introduced by the Bourbaki group was the agreement to make mathematics a separate social subsystem. This method defines a function of the subsystem "mathematics" that fulfils the criteria Luhmann (1984) explains. The consequence was a functional differentiation, a separation from the other sciences.

As shown in this article, the foundations for mathematics to become a subsystem of the science system were laid in the 19th century. The agreement on a specific mathematical method to decide the truth of theories was the birth of the social subsystem called "mathematics".

Finally, the implications for educators of adults learning mathematics are that they should know more about the history of mathematics and, from this article, keep in mind that the development of mathematics is strongly influenced by decisions made by mathematicians. Following the interpretation guided by Luhmann's theory, it is significant that these decisions are made according to what "should" happen.

References

Aristotele (1948). *Organon*. Leipzig: Reclam.

Becker, O. (1927). Mathematische Existenz. In *Jahrbuch für Philosophie und phänomenologische Forschung 8*. Halle a. S., Max Niemeyer Verlag.

Bernays, P. (1930–1931). Die Philosophie der Mathematik und die Hilbertsche Beweistheorie, in: *Blätter für deutsche Philosophie 4*, 326–67. [English translation in Mancosu (1998a, 234–265).]

Du Bois-Reymond, P. (1875). Versuch einer Classification der willkürlichen Functionen reeller Argumente nach ihren Änderungen in den kleinsten Intervallen. *Crelles Journal* 79.

Boole, G. (1854/1958). *An investigation of the laws of thought, on which are founded the mathematical theories of logic and probability*. New York: Dover.

Bourbaki, N. (1934). *Elements de mathematique*, Paris: Springer-Verlag.

Bourbaki, N. (1971). *Elemente der Mathematikgeschichte*/. Göttingen: Vandenhoeck & Ruprecht.

Brouwer, L.E.J. (1907). *Over de Grondslagen der Wiskunde*. Akademische Prüfschrift, Amsterdam, Leipzig: Maas & van Suchtelen.

Burali-Forti, C. (1897). Una questione sui numeri transfiniti. *Rendiconti del Circolo Matematico di Palermo* 11, Circolo Matematico.

Cantor, G. (1932). *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts* (hg. v. E. Zermelo). Berlin: Springer-Verlag

Cartan, H. (1959). Nikolas Bourbaki und die heutige Mathematik. *Hefte der Arbeitsgemeinschaft für Forschung des Landes NRW*, Bd. 76, pp. 1–27. Köln, Opladen: Westdeutscher Verlag.

Cauchy, A.-L. (1821). Co*urs d'Analyse de l'Ecole Royale Polytechnique*. Paris: Imp. Roy. Debure Frères.

David, E. (1984). Toward renewing a threatened resource. Findings and recommendations of the Ad Hoc Committee on Resources for the Mathematical Society. *Notices of the American Mathematical Society*, *31*(2), 141–145.

Ferber C. (1956). *Die Entwicklung des Lehrkörpers der deutschen Universitäten und Hochschulen 1864 -, 1954*. Göttingen: Vandenhoeck & Ruprecht.

Fraenkel, A. (1927). *Zehn Vorlesungen über die Grundlagen der Mengenlehre*, Leipzig, Berlin (Nachdruck Darmstadt, 1972): Wissenschaftliche Buchgesellschaft.

Frege, G. (1884/1961). *Die Grundlagen der Arithmetik*. Neudruck Darmstadt: Wissenschaftliche Buchgesellschaft.

Frege, G. (1993, 1903). *Grundgesetze der Arithmetik - begriffsschriftlich abgeleitet*, Bd. I Jena 1893, Bd. II Jena 1903: Verlag Hermann Pohle.

Gödel, K. (1931). Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I. *Monatshefte für Mathematik und Physik*, Bd. 38, pp. 173–198.

Heyting, A. (1934). *Mathematische Grundlagenforschung – Intuitionismus – Beweistheorie*, Berlin (Neudruck Berlin, Heidelberg, New York, 1974): Springer Verlag.

Hilbert, D. (1899). *Grundlagen der Geometrie*, zit. n. 11. Aufl., Stuttgart, 1968: Teubner.

Hilbert, D. (1922). Neubegründung der Mathematik, Erste Mitteilung. *Abhandlungen aus dem mathematischen Seminar der Hamburger Universität*, Bd. 1, pp. 157–177.

Hilbert, D. (1923). Die logischen Grundlagen der Mathematik, in: Ders.: *Gesammelte Abhandlungen*, Bd. III, pp. 151–165. Berlin: Springer Verlag.

Hilbert, D. (1925). Über das Unendliche. *Mathematische Annalen, 95*, 161–190.

Inhetveen, H. (1976). *Die Reform des gymnasialen Mathematikunterrichts zwischen 1890 und 1914. Eine sozioökonomische Analyse*. Bad Heilbrunn: Verlag Julius Klinkhardt.

Kant, I. (1781). *Kritik der reinen Vernunft*. Riga, (Leipzig, 1979): Reclam Verlag.

Klein, F. (1926). *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert*, Teil 1 und 2 (hg. v. R. Courant u. O. Neugebauer). Berlin, 1926, Nachdruck Berlin, Heidelberg, New York, 1979: Springer Verlag.

Koch, J.F.W. (1839 & 1840). *Die preußischen Universitäten*, Bd. I und II. Berlin: Mittler Verlag.

Lobatschewski, N.I. (1898). *Zwei geometrische Abhandlungen*. *Aus dem Russischen übersetzt von F. Engel*. Leipzig: Teubner Verlag.

Lorenzen, P. (1958). Die klassische Analysis als konstruktive Theorie. In P. Lorenzen (Ed.) *Methodisches Denken* (pp. 104–119). Frankfurt: Suhrkamp.

Lorenzen, P. Schwemmer, O. (1975). *Konstruktive Logik, Ethik und Wissenschaftstheorie*, 2. Aufl. Mannheim, Wien, Zürich: Bibliographisches Institut.

Lorey, W. (1913). Zu der Nachricht über den ersten stud. Philosophy. *Neue Jahrbücher für Pädagogik* (pp. 18–27). Leipzig: Teubner.

Lorey, W. (1916). *Das Studium der Mathematik an den deutschen Universitäten seit Anfang des 19 Jahrhunderts*. Leipzig und Berlin: Teubner Verlag.

Luhmann, N. (1970). Selbststeuerung der Wissenschaft. Ders.: *Soziologische Aufklärung*, pp. 232–252. Köln und Opladen: Westdeutscher Verlag.

Luhmann, N. (1984). *Soziale Systeme*. Frankfurt: Suhrkamp Verlag.

Maasz, J. (1985). *Mathematik als soziales System* (Diss.), Universität Essen GH.

Maasz, J. (1987). Anmerkungen zur “Encyklopädie der mathematischen Wissenschaften” aus wissenschaftssoziologischer Sicht. *Beiträge zum Mathematikunterricht* (pp. 142–146). Bad Salzdetfurth: Franzbecker Verlag.

Maasz, J. (1988). *Mathematik als soziales System. Geschichte und Perspektiven der Mathematik aus systemtheoretischer Sicht*. Deutscher Studien-Verlag, Weinheim: Beltz Verlag.

Maaß, J., & Schlöglmann, W. (1988). The mathematical world in the black box-significance of the black box as a medium of mathematizing. *Cybernetics and Systems: An International Journal, 19*, 295–309.

Mancosu, P. (Ed.) (1998). *From Brouwer to Hilbert. The debate on the foundations of mathematics in the 1920s*. Oxford: Oxford University Press.

von Neumann, J. (1931). Die formalistische Grundlegung der Mathematik. *Erkenntnis 2*, 116–134.

von Neumann, J. (1947). The mathematician. In R. B. Heywood (Ed.), *The works of mind* (pp. 180–196). Chicago: University of Chicago Press.

Pfetsch, F. R. (1974). *Zur Entwicklung der Wissenschaftspolitik in Deutschland 1750 – 1914*. Berlin: Duncker & Humblot Verlag.

Quine, W. V. (1955). On Frege’s way out. *Mind 64*, 145–159.

Rusawin, G. I. (1968). *Über die Natur der mathematischen Erkenntnis* (russ.). Moskau, Übersetzung: Berlin (DDR): Deutscher Verlag der Wissenschaften.

Russell, B. (1901). Die Mathematik und die Metaphysiker. In *Kursbuch* 8 (pp. 8–25). Frankfurt: Zweitausendeinsverlag.

Russell, B. (1903). *The principles of mathematics*. London: Cambridge University Press.

Schubring, G. (1983). Seminar – Institut – Fakultät: Die Entwicklung der Ausbildungsformen und ihrer Institutionen in der Mathematik, in: Interdisziplinäres Zentrum für Hochschuldidaktik der Universität Bielefeld (Hrsg.). *Diskussionsbeiträge zur Ausbildungsforschung und Studienreform* Heft 1.

Spalt, D. (1981). *Vom Mythos der mathematischen Vernunft. Eine Archäologie zum Grundlagenstreit der Analysis oder Dokumentation einer vergeblichen Suche nach der Einheit der mathematischen Vernunft*. Darmstadt: Wissenschaftliche Buchgesellschaft.

Stegmüller, W. (1978). *Hauptströmungen der Gegenwartsphilosophie*. Bd. 1, 6. Aufl. Stuttgart: Kröner Verlag.

Steck, M. (Ed.) (1941). Unbekannte Briefe Freges über die Grundlagen der Geometrie und Antwortbrief Hilberts an Frege. *Sitzungsberichte der Heidelberger Akademie der Wissenschaften, Mathematisch–Naturwissenschaftliche Klasse*. Heidelberg: Kommissionsverlag der Weiß’schen Universitätsbuchhandlung Heidelberg.

Struik, D. J. (1976). *Abriß der Geschichte der Mathematik*. Berlin (DDR): Deutscher Verlag der Wissenschaften.

Thiel, C. (1972). *Grundlagenkrise und Grundlagenstreit*. Meisenheim: Verlag Anton Hein.

Tobies, R. (1985). *Die gesellschaftliche Stellung deutscher mathematischer Organisationen und ihre Funktion bei der Veränderung der gesellschaftlichen Wirksamkeit der Mathematik* (1871–1933), Diss (B), Leipzig.

Tobies, R. (1989). On the contribution of mathematical societies to promoting applications of mathematics in Germany. In D.E. Rowe & J. McCleary (Eds.), *The history of modern mathematics, vol II: Institutions and applications* (pp. 223–248). Boston, San Diego, New York: Academic Press.

Wang, H. (1954). The formalization of mathematics. *Journal of Symbolic Logic*, *19*, 1–16.

Whitehead, A. N., & Russell, B. (1910, 1912, 1913). *Principia Mathematica*, Bks. I – III. Cambridge: Cambridge University Press.

Wußing, W. (1983). *Biographien bedeutender Mathematiker*, 3. Aufl. Berlin: Deutscher Verlag der Wissenschaften.

Zermelo, E. (1908). Untersuchungen über die Grundlagen der Mengenlehre I. *Mathematische Annalen 65,* 261–281.

[1] All translations from the German are by the author.